

First results from simulations of supersymmetric lattices

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ABSTRACT: We conduct the first numerical simulations of lattice theories with exact supersymmetry arising from the orbifold constructions of [1, 2, 3]. We consider the $\mathcal{Q} = 4$ theory in $D = 0, 2$ dimensions and the $\mathcal{Q} = 16$ theory in $D = 0, 2, 4$ dimensions. We show that the $U(N)$ theories do not possess vacua which are stable non-perturbatively, but that this problem can be circumvented after truncation to $SU(N)$. We measure the distribution of scalar field eigenvalues, the spectrum of the fermion operator and the phase of the Pfaffian arising after integration over the fermions. We monitor supersymmetry breaking effects by measuring a simple Ward identity. Our results indicate that simulations of $\mathcal{N} = 4$ super Yang-Mills may be achievable in the near future.

KEYWORDS: Lattice gauge theory; dynamical fermion simulations; topological field theory; supersymmetry.

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1. Introduction

The study of supersymmetric theories on lattices has a long history – see the review [4] and references therein. Recently there has been a resurgence of interest in the field with successful constructions of lattice theories which keep intact a subalgebra of the full supersymmetry algebra [5, 6, 7].

In this paper we will be concerned with specific discretizations of the $\mathcal{Q} = 4$ and $\mathcal{Q} = 16$ supercharge Yang-Mills theories in a variety of dimensions. The lattice actions we employ were first derived using orbifold/deconstruction techniques in [1, 2, 3]. Recently, it was shown how to recover them by discretization of a twisted version of the target super Yang-Mills theories [8]¹. Other proposals for lattice actions based on twisting can be found in [13, 14, 15, 16, 17, 18]. We will use the language of the twisted constructions in this paper.

¹The connection between topological twisting and orbifold constructions had been anticipated earlier in [9, 10, 11] and has since been generalized in [12]

The key feature of these actions is that, in addition to gauge invariance, they retain one or more exact supersymmetries at non vanishing lattice spacing. Since this feature is already sufficient to pair each bosonic state with a fermionic state of the same energy it is expected that these models may flow to the target continuum theory with a minimum of fine tuning as the lattice spacing is sent to zero.

In this paper, we show that the vacua of these lattice theories with $U(N)$ gauge symmetry exhibit a non-perturbative instability associated to a runaway trace mode of the scalar fields. We show that this problem can be evaded if the gauge group is truncated to $SU(N)$ at the price of a mild breaking of the exact supersymmetry. In two dimensions and for the $\mathcal{Q} = 16$ theory we show that, nevertheless, supersymmetry is restored without fine tuning in the continuum limit. The situation in the two-dimensional $\mathcal{Q} = 4$ theory is more problematic on account of the observed strong fluctuations in the phase of the Pfaffian that results from the fermion integration. These fluctuations lead to large statistical errors in all observables making it difficult to draw definite conclusions.

In contrast we observe rather small phase fluctuations for the $\mathcal{Q} = 16$ theory with gauge group $SU(2)$ in both two and four dimensions for the small lattices used in this study.

The organization of the paper is as follows; we first summarize the lattice actions with which we are concerned, describe some of the details of the simulation algorithms that are employed and then summarize our numerical results for the case of $\mathcal{Q} = 4$ and $\mathcal{Q} = 16$ supercharge theories in dimensions zero, two and four. We end with a discussion and outlook for the future.

2. Lattice actions

2.1 $\mathcal{Q} = 4$ theory in two dimensions

The field content of the lattice theory comprises a multiplet of p-form fermions $(\eta, \psi_\mu, \chi_{12})$ distributed over sites and links of the lattice together with a complexified Wilson gauge link \mathcal{U}_μ . These fields transform under a scalar supersymmetry as follows:

$$\begin{aligned}\mathcal{Q} \mathcal{U}_\mu &= \psi_\mu \\ \mathcal{Q} \psi_\mu &= 0 \\ \mathcal{Q} \bar{\mathcal{U}}_\mu &= 0 \\ \mathcal{Q} \chi_{\mu\nu} &= \mathcal{F}_{\mu\nu}^\dagger \\ \mathcal{Q} \eta &= d \\ \mathcal{Q} d &= 0\end{aligned}\tag{2.1}$$

Furthermore, as was shown in [8] the action of the theory can be written in a \mathcal{Q} -exact form

$$S = \kappa \mathcal{Q} \left[\sum_{\mathbf{x}} \text{Tr} \left(\chi_{\mu\nu} \mathcal{F}_{\mu\nu} + \eta \bar{\mathcal{D}}_\mu^{(-)} \mathcal{U}_\mu - \frac{1}{2} \eta d \right) \right]\tag{2.2}$$

where κ is a dimensionless bare coupling. The decomposition of the fermions into p-forms and the appearance of a scalar supersymmetry arises as a consequence of the twisting procedure which was described in detail in [19, 20].

Notice that this supersymmetry is nilpotent making the supersymmetric invariance of the the lattice action in eqn. 2.2 manifest. In this formulation the gauge links are *non-unitary* matrices of the form $U_\mu(x) = e^{A_\mu(x) + iB_\mu(x)}$ with $\mu = 1, 2$. The imaginary parts of the gauge fields generate the scalar fields of theory in the continuum limit as was shown in [8]². The gauge covariant difference operators appearing in eqn. 2.2 are defined by

$$\mathcal{D}_\mu^{(+)} f_\nu(\mathbf{x}) = \mathcal{U}_\mu(\mathbf{x}) f_\nu(\mathbf{x} + \mathbf{e}_\mu) - f_\nu(\mathbf{x}) \mathcal{U}_\mu(\mathbf{x} + \mathbf{e}_\nu) \quad (2.3)$$

$$\overline{\mathcal{D}}_\mu^{(-)} f_\mu(\mathbf{x}) = f_\mu(\mathbf{x}) \overline{\mathcal{U}}_\mu(\mathbf{x}) - \overline{\mathcal{U}}_\mu(\mathbf{x} - \mathbf{e}_\mu) f_\mu(\mathbf{x} - \mathbf{e}_\mu) \quad (2.4)$$

where the unit lattice vectors are $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$ and the lattice field strength $\mathcal{F}_{\mu\nu}$ is given by

$$\mathcal{F}_{\mu\nu} = \mathcal{D}_\mu^{(+)} \mathcal{U}_\nu(\mathbf{x}) - \mathcal{U}_\mu(\mathbf{x}) \mathcal{U}_\nu(\mathbf{x} + \boldsymbol{\mu}) - \mathcal{U}_\nu(\mathbf{x}) \mathcal{U}_\mu(\mathbf{x} + \boldsymbol{\nu}) \quad (2.5)$$

In general the lattice fields are associated to links of the lattice and they transform correspondingly under gauge transformations:

$$\begin{aligned} \eta(\mathbf{x}) &\rightarrow G(\mathbf{x}) \eta(\mathbf{x}) G^\dagger(\mathbf{x}) \\ \psi_\mu(\mathbf{x}) &\rightarrow G(\mathbf{x}) \psi_\mu(\mathbf{x}) G^\dagger(\mathbf{x} + \mathbf{e}_\mu) \\ \chi_{\mu\nu}(\mathbf{x}) &\rightarrow G(\mathbf{x} + \mathbf{e}_\mu + \mathbf{e}_\nu) \chi_{\mu\nu}(\mathbf{x}) G^\dagger(\mathbf{x}) \\ \mathcal{U}_\mu(\mathbf{x}) &\rightarrow G(\mathbf{x}) \mathcal{U}_\mu(\mathbf{x}) G^\dagger(\mathbf{x} + \mathbf{e}_\mu) \\ \overline{\mathcal{U}}_\mu(\mathbf{x}) &\rightarrow G(\mathbf{x} + \mathbf{e}_\mu) \overline{\mathcal{U}}_\mu(\mathbf{x}) G^\dagger(\mathbf{x}) \end{aligned} \quad (2.6)$$

Notice that this choice of link and orientation for the twisted lattice fields maps exactly into their r-charge assignments in the orbifolding approach [1]. Indeed, the lattice actions derived by the two methods are identical.

The final lattice action takes the form

$$S = \sum_{\mathbf{x}} \text{Tr} \left(\mathcal{F}_{\mu\nu}^\dagger \mathcal{F}_{\mu\nu} + \frac{1}{2} \left(\overline{\mathcal{D}}_\mu^{(-)} \mathcal{U}_\mu \right)^2 - \chi_{\mu\nu} \mathcal{D}_{[\mu}^{(+)} \psi_{\nu]} - \eta \overline{\mathcal{D}}_\mu^{(-)} \psi_\mu \right) \quad (2.7)$$

where we have integrated out the auxiliary field d .

It is also possible to consider dimensional reductions of this model to lower dimensions. For example we will show results obtained from simulations of the $D = 0$ matrix model that results from this lattice theory after dropping all dependence on lattice coordinates \mathbf{x} . The dimensionless lattice coupling κ in reduced dimension D is given by

$$\kappa = \frac{N}{2\lambda} \left(\frac{L}{\beta} \right)^{(4-D)} \quad (2.8)$$

where $\lambda = g^2 N$ is the 't Hooft coupling, β is the physical extent of the system and L the lattice length.

²We use an antihermitian basis for the generators

2.2 $\mathcal{Q} = 16$ theory in four dimensions

Remarkably, the $\mathcal{Q} = 16$ supercharge theory in four dimensions also possesses a \mathcal{Q} -exact term in its action of precisely the same form as its two dimensional cousin:

$$S = \kappa \mathcal{Q} \left[\sum_{\mathbf{x}} \text{Tr} \left(\chi_{ab} \mathcal{F}_{ab} + \eta \overline{\mathcal{D}}_a^{(-)} \mathcal{U}_a - \frac{1}{2} \eta d \right) \right] \quad (2.9)$$

where the indices now run $a, b = 1 \dots 5$ and the sixteen fermions of $\mathcal{N} = 4$ super Yang-Mills are now built from one scalar, five vectors and the ten components of the antisymmetric tensor χ_{ab} . The real parts of $U_\mu, \mu = 1 \dots 4$ yield the usual four dimensional gauge field while U_5 and the remaining imaginary components of U_μ yield the expected six scalar fields of $\mathcal{N} = 4$ super Yang-Mills [8].

The action of the nilpotent, scalar supersymmetry is the same as before

$$\begin{aligned} \mathcal{Q} \mathcal{U}_a &= \psi_a \\ \mathcal{Q} \overline{\mathcal{U}}_a &= 0 \\ \mathcal{Q} \psi_a &= 0 \\ \mathcal{Q} \chi_{ab} &= \mathcal{F}_{ab}^\dagger \\ \mathcal{Q} \eta &= d \\ \mathcal{Q} d &= 0 \end{aligned} \quad (2.10)$$

In addition a new \mathcal{Q} -closed term is needed of the form

$$S_{\text{closed}} = -\frac{\kappa}{8} \sum_{\mathbf{x}} \text{Tr} \epsilon_{abcde} \chi_{de}(\mathbf{x} + \mathbf{e}_a + \mathbf{e}_b + \mathbf{e}_c) \mathcal{D}_c^{(-)} \chi(\mathbf{x} + \mathbf{e}_c) \quad (2.11)$$

which is zero by virtue of an exact Bianchi identity satisfied by the lattice field strength

$$\epsilon_{abcde} D_c^{(+)} \mathcal{F}_{de} = 0 \quad (2.12)$$

Gauge invariance of this term requires $\sum_a^5 \mathbf{e}_a = 0$. This condition is satisfied if the basis vectors are taken to be $\mathbf{e}_a^i = \delta_j^i$ with $a = 1 \dots 4$ and $\mathbf{e}_5 = (-1, -1, -1, -1)$ corresponding to their r-charge assignments in the orbifold construction [3].

Again, the link assignments of the fields can be summarized by specifying the variation of the fields under gauge transformations

$$\begin{aligned} \eta(\mathbf{x}) &\rightarrow G(\mathbf{x}) \eta(\mathbf{x}) G^\dagger(\mathbf{x}) \\ \psi_a(\mathbf{x}) &\rightarrow G(\mathbf{x}) \psi_a(\mathbf{x}) G(\mathbf{x} + \mathbf{e}_a) \\ \chi_{ab}(\mathbf{x}) &\rightarrow G(\mathbf{x} + \mathbf{e}_a + \mathbf{e}_b) \chi_{ab}(\mathbf{x}) G^\dagger(\mathbf{x}) \\ \mathcal{U}_a(\mathbf{x}) &\rightarrow G(\mathbf{x}) \mathcal{U}_a(\mathbf{x}) G^\dagger(\mathbf{x} + \mathbf{e}_a) \\ \overline{\mathcal{U}}_a(x) &\rightarrow G(\mathbf{x} + \mathbf{e}_a) \overline{\mathcal{U}}_a(\mathbf{x}) G^\dagger(\mathbf{x}) \end{aligned} \quad (2.13)$$

$$(2.14)$$

The action of the covariant difference operators appearing in the action take the same form as for the $\mathcal{Q} = 4$ theory eqn. 2.4. In the naive continuum limit it is possible to show that this lattice theory reduces to the Marcus topological twist of $\mathcal{N} = 4$ super Yang-Mills [21] (in modern parlance the GL-twist [22]) and hence in flat space is fully equivalent to the usual continuum theory.

Again, the model can be reduced to lower dimensions by simply taking the fields to be independent of certain lattice coordinates and we will show data for the model in both zero, two and four dimensions. The bare coupling κ is again given by eqn. 2.8.

3. Simulation details

Since both the $\mathcal{Q} = 4$ and $\mathcal{Q} = 16$ supercharge theories have such a similar structure they can be simulated using a single Monte Carlo code in which the lattice geometry, gauge group and number of supercharges are input as parameters. The dynamical twisted fermions are handled exactly and efficiently using the RHMC algorithm [23]. For completeness we summarize the main features of this algorithm as applied to the simulation of supersymmetric lattices here.

If we denote the set of twisted fermions by the field $\Psi = (\eta, \psi_\mu, \chi_{\mu\nu})$ we first introduce a parallel pseudofermion field Φ with action

$$S_{\text{PF}} = \Phi^\dagger (M^\dagger M)^{-\frac{1}{4}} \Phi \quad (3.1)$$

where $M = M(\mathcal{U}, \mathcal{U}^\dagger)$ is the antisymmetric twisted lattice fermion operator given, for example, in eqn. 2.7³.

Integrating over the fields Φ will then yield (up to a possible phase) the Pfaffian of the operator $M(\mathcal{U}, \mathcal{U}^\dagger)$ as required. The fractional power is approximated by the partial fraction expansion

$$\frac{1}{(M^\dagger M)^{\frac{1}{4}}} = \alpha_0 + \sum_{i=1}^P \frac{\alpha_i}{M^\dagger M + \beta_i} \quad (3.2)$$

where the coefficients $\{\alpha_i, \beta_i\}$ are evaluated offline using the Remez algorithm to minimise the error in some interval (ϵ, A) . Typically we have used $P = 15$ which yields a fractional error of 0.00001 for the interval $0.0000001 \rightarrow 1000.0$ which conservatively covers the range we are interested in.

Following the standard procedure we introduce momenta $(p_{\mathcal{U}}, p_F)$ conjugate to the coordinates (\mathcal{U}, Φ) and evolve the coupled system using a discrete time leapfrog algorithm according to the classical Hamiltonian $H = S_B + S_{\text{PF}} + p_{\mathcal{U}} \bar{p}_{\mathcal{U}} + p_{\Phi} \bar{p}_{\Phi}$. One step of the discrete time

³The antisymmetry is guaranteed if the fermion action is rewritten as the sum of the original terms plus their lattice transposes

update is given by

$$\delta p_{\mathcal{U}} = \frac{\delta t}{2} \bar{f}_{\mathcal{U}} \quad (3.3)$$

$$\delta p_{\Phi} = \frac{\delta t}{2} \bar{f}_{\Phi} \quad (3.4)$$

$$\delta \mathcal{U} = \left(e^{\delta t p_{\mathcal{U}}} - I \right) \mathcal{U} \quad (3.5)$$

$$\delta \Phi = \delta t p_{\Phi} \quad (3.6)$$

$$\delta p_{\mathcal{U}} = \frac{\delta t}{2} \bar{f}_{\mathcal{U}} \quad (3.7)$$

$$\delta p_{\Phi} = \frac{\delta t}{2} \bar{f}_{\Phi} \quad (3.8)$$

where the forces $f_{\mathcal{U}}$ and f_{Φ} are given by

$$f_{\mathcal{U}} = -\frac{\delta S}{\delta \mathcal{U}} \quad (3.9)$$

$$f_{\Phi} = -\frac{\delta S}{\delta \Phi} \quad (3.10)$$

and the bar denotes complex conjugation. Using the partial fraction expansion given in eqn. 3.2 these forces take the form

$$f_{\mathcal{U}} = \sum_{i=1}^P \alpha_i \left[\bar{t}_i \frac{\delta M}{\delta \mathcal{U}} s_i + \overline{\left(\bar{t}_i \frac{\delta M}{\delta \bar{\mathcal{U}}} s_i \right)} \right] \quad (3.11)$$

$$f_{\Phi} = -\alpha_0 \bar{\Phi} - \sum_{i=1}^P \alpha_i \bar{s}_i \quad (3.12)$$

$$(3.13)$$

where

$$(M^{\dagger} M + \beta_i) s_i = \Phi \quad (3.14)$$

$$t_i = M s_i \quad (3.15)$$

The latter set of sparse linear equations is solved using a multimass CG-solver [24] which allows for the simultaneous solution of all P systems in a single CG solve.

At the end of one such classical trajectory the final configuration is subjected to a standard Metropolis test based on the Hamiltonian H . The symplectic and reversible nature of the discrete time update is then sufficient to allow for detailed balance to be satisfied and hence expectation values are independent of δt . After each such trajectory the momenta are refreshed from the appropriate gaussian distribution as determined by H which renders the simulation ergodic.

In the work reported here we have employed only periodic boundary conditions which preserve supersymmetry. Thermal boundary conditions are also interesting as they allow for

exploration of dualities between string and gauge theory [25, 26, 27, 28, 29, 30] and dynamical supersymmetry breaking [31, 32].

Our measurements concentrate, in part, on local observables such as the eigenvalues of $\mathcal{U}_\mu^\dagger(x)\mathcal{U}_\mu(x)$ and the bosonic action $\langle S_B(\mathcal{U}) \rangle$. The former yields, in the continuum limit, the distribution of eigenvalues of the scalar fields $\mathcal{U}_\mu^\dagger\mathcal{U}_\mu - I = 2B_\mu + \dots$ and hence gives information on the quantum moduli space. The latter observable is related to an exact supersymmetric Ward identity and can be evaluated analytically which provides both a useful check on our code and measures the magnitude of supersymmetry breaking effects. The analytic argument that determines the value of $\langle S_B \rangle$ is simple. Consider first the $\mathcal{Q} = 4$ supercharge theory and write down an expression for the mean action $\langle S \rangle$

$$\langle S \rangle = -\frac{\partial \ln Z}{\partial \kappa} = \langle \mathcal{Q}\Lambda \rangle = 0 \quad (3.16)$$

where the last result follows from \mathcal{Q} -exact nature of the twisted action and shows that the vanishing mean action is the consequence of a simple \mathcal{Q} -Ward identity. This argument needs a minor modification for the $\mathcal{Q} = 16$ supercharge theory which contains also a \mathcal{Q} -closed term. However, it is straightforward to show that a simple rescaling of the field $\chi_{\mu\nu} \rightarrow \sqrt{\kappa}\chi_{\mu\nu}$ removes the κ dependence of this term so that once again $\frac{\partial \ln Z}{\partial \kappa} = 0$ as a consequence of \mathcal{Q} -supersymmetry⁴.

The result $\langle S \rangle = \langle S_B \rangle + \langle S_F \rangle = 0$ can be translated into an exact result for the bosonic action since the fermions appear quadratically in the action and hence $\langle S_F \rangle$ can be evaluated by a simple scaling argument. In the case of $\mathcal{Q} = 4$ supercharges one finds

$$\kappa \langle S_B \rangle = \frac{3}{2}N_G V \quad (3.17)$$

where N_G is the number of generators of the group and V is the number of lattice points. One might have naively expected a factor of 4 representing the four twisted fermions of the $\mathcal{Q} = 4$ theory rather than the factor of 3 that is present in this expression - the discrepancy arises as a consequence of integrating out the auxiliary field d which effectively removes the contribution of one fermion to $\langle S_B \rangle$.

The $\mathcal{Q} = 16$ theory is a little more involved. To evaluate $\langle S_F \rangle$ by a scaling argument requires an additional rescaling of both η and ψ_μ by a factor of $\sqrt{\kappa}$. This results in an additional multiplicative factor of $\kappa^{6N_G V/2}$ in the measure. The final result for $\langle S_B \rangle$ is then

$$\kappa \langle S_B \rangle = \frac{9}{2}N_G V \quad (3.18)$$

where the factor arising in the numerator is now composed from $9 = 16 - 6 - 1$.

A faster way to derive these results relies simply on the coupling constant independence of the free energy; the bosonic action can then be evaluated in the weak coupling limit where

⁴Another way to see this is to realize that the partition function for periodic boundary conditions is just the Witten index and hence does not depend on the coupling constant

the theory is quadratic in the bosons and equipartition holds; the bosonic action then simply counts the number of degrees of freedom.

For the small systems we have examined in this paper we have also measured the Pfaffian and the spectrum of the fermion operator $M(\mathcal{U}, \overline{\mathcal{U}})$. The Pfaffian computation is carried out by using a variant of Gaussian elimination with full pivoting to transform the $2n \times 2n$ dimensional antisymmetric matrix M into the canonical form

$$\begin{pmatrix} 0 & \lambda_1 & 0 & 0 & \dots \\ -\lambda_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \lambda_2 & \dots \\ 0 & 0 & -\lambda_2 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \end{pmatrix} \quad (3.19)$$

Then

$$\text{Pf}(M) = \prod_{i=1}^n \lambda_i \quad (3.20)$$

As can be seen by examining eqn. 3.1 our simulations generate the *phase quenched* ensemble defined by $|Pf(M)|$. As usual we can always compensate for neglecting any phase $e^{i\alpha(\mathcal{U})}$ by re-weighting all observables $O(\mathcal{U})$ by the phase factor according to the simple rule

$$\langle O \rangle = \frac{\langle O(\mathcal{U}) e^{i\alpha(\mathcal{U})} \rangle_{\alpha=0}}{\langle e^{i\alpha(\mathcal{U})} \rangle_{\alpha=0}} \quad (3.21)$$

Where necessary we have reweighted our results accordingly.

4. Zero dimensions

4.1 Vacuum instability for $U(N)$ theories

The construction we have described is strictly valid only for $U(N)$ theories. This can be seen in a variety of ways; taking the trace of the first line of the \mathcal{Q} -variations in eqn. 2.10 is inconsistent if the fields are restricted to the traceless generators. Similarly the difference operators employed in the lattice action when applied to a traceless field generically yield a field with non-zero trace. Thus we initially focus on simulations of the $U(N)$ theories. Unfortunately we will see that these theories have a non-perturbative instability; the trace mode of the scalar fields rapidly runs off to (negative) infinity.

Figure 1 shows this explicitly with a plot of the Monte Carlo history of the eigenvalues of $\mathcal{U}_\mu^\dagger(x) \mathcal{U}_\mu(x)$ for an $SU(2)$ model with $\mathcal{Q} = 4$ supersymmetries reduced to zero dimensions. Clearly one of the eigenvalues is driven to zero after a finite simulation time. Using the representation

$$\mathcal{U}_\mu(x) = e^{A_\mu(x) + iB_\mu(x)} \quad (4.1)$$

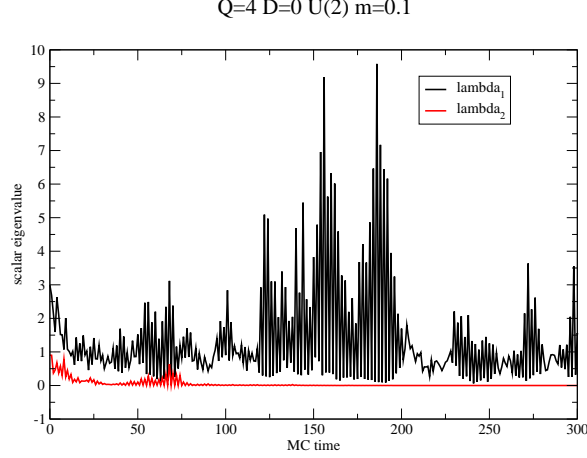


Figure 1: Eigenvalues of $\mathcal{U}_\mu^\dagger \mathcal{U}_\mu$ versus MC time. Gauge group $SU(2)$, $Q = 4$ and $D = 0$

m	S_B	S_B^{exact}	$\frac{1}{2} \text{Tr} (\mathcal{U}_\mu^\dagger \mathcal{U}_\mu)$
0.01	2.65(6)	6.0	0.45(2)
0.1	2.40(8)	6.0	0.57(6)
0.5	2.99(7)	6.0	0.38(2)

Table 1: Mass dependence of $\frac{1}{2} \text{Tr} \mathcal{U}_\mu^\dagger \mathcal{U}_\mu$

it is easy to see that $\det(M(x)) = e^{2\sqrt{N}B_\mu^0(x)}$ where $B_\mu^0(x)$ is the trace component of the scalar field⁵. The presence of a zero eigenvalue leads to a vanishing determinant and clearly is associated with a limit in which the trace mode $B_\mu^0 \rightarrow -\infty$. In this situation the fluctuations in the gauge links are never small and no limit exists in which the lattice model approximates the correct continuum theory. Furthermore, we have observed that no simple scalar mass term as advocated in [1] is able to save the situation. Table 1 shows the value of $\langle \frac{1}{N} \text{Tr} \mathcal{U}_\mu^\dagger(x) \mathcal{U}_\mu(x) \rangle$ obtained from a series of runs in which the lattice action is supplemented with the additional term

$$\Delta S = m^2 \sum_x \left(\mathcal{U}_\mu^\dagger(x) \mathcal{U}_\mu(x) - I \right)^2 \quad (4.2)$$

The first column gives the mean bosonic action as compared, in the second column, to its exact value as predicted by eqns. 3.17 and 3.18. Clearly, the \mathcal{U}_μ fields deviate strongly from the unit matrix for all values of the mass parameter m and furthermore supersymmetry is also badly broken. In all these cases we also observe a zero eigenvalue in M independent of the mass. These conclusions survive for other gauge groups $U(N)$, values of the supercharge Q and dimension of the model.

⁵Clearly one zero eigenvalue is the minimum required for consistency with a vanishing determinant – we have also encountered runs where some or all of the $\mathcal{U}_\mu^\dagger \mathcal{U}_\mu$ matrices have more than one zero eigenvalue

It is straightforward to see why this occurs; the bosonic action evaluated on an arbitrary $\{\mathcal{U}_\mu(x)\}$ configuration is generically lowered by a global shift in the trace mode $B_\mu^0(x) \rightarrow B_\mu^0(x) - c$ with positive c^6 . Explicitly,

$$S_B(\mathcal{U}_\mu(x)e^{-cI}) = e^{-4c}S_B(\mathcal{U}_\mu(x)) \quad (4.3)$$

since e^{-cI} commutes with all terms in the bosonic action (each of which contains a product of four \mathcal{U}_μ matrices). Unlike in the continuum the penalty from the mass term is finite even in the limit $B_\mu^0 \rightarrow -\infty$ and entropic effects appear to drive the system far from the point $B_\mu^0 = 0$. Notice that this effect occurs only for the exponential parametrization of the complex \mathcal{U}_μ matrices used here and given in eqn. 4.1. The prescription utilized in the original orbifold constructions used instead the decomposition

$$\mathcal{U}_\mu = I + A_\mu + iB_\mu \quad (4.4)$$

which is not subject to this same instability.

One way to evade these problems immediately presents itself; simply set the trace mode to zero by truncating the lattice model to the special unitary group $SU(N)$. Notice that this truncation preserves supersymmetry if the (complexified) gauge fields lie infinitesimally close to the identity $\mathcal{U}_\mu(x) = I + aA_\mu(x) + \dots$ which should happen in the continuum limit. Presumably the restriction to $SU(N)$ is also irrelevant in the large N limit. However, for non-zero lattice spacing and finite N it will lead to supersymmetry breaking effects which we examine in the next section.

4.2 Vacuum structure for $SU(N)$ theories

4.2.1 $\mathcal{Q} = 4$ supercharges

Once the truncation to $SU(N)$ is carried out we observe no instability; the eigenvalues of $\mathcal{U}_\mu^\dagger \mathcal{U}_\mu$ cluster around unity for any gauge group, dimension or number of supercharges. Furthermore the width of this eigenvalue distribution decreases as the bare lattice coupling increases which is a necessary condition for the lattice model to approximate the target theory in the continuum limit.

Nevertheless, after this truncation we might expect to see supersymmetry breaking effects and we have probed for this by examining the bosonic action. As we have argued earlier the expectation value of this term can be derived exactly as a consequence of the twisted scalar supersymmetry. Deviations from the exact value hence measure supersymmetry breaking. Table 2 shows results from a simulation of the $\mathcal{Q} = 4$ model with $SU(2)$ gauge group in the matrix model limit⁷. While a small, statistically significant deviation is seen for strong coupling this disappears for larger κ . The limit $\kappa \rightarrow \infty$ corresponds to the continuum limit $a \rightarrow 0$ for theories in $1 \leq D < 4$. A similar pattern seen in table 3 for the same model with group $SU(3)$.

⁶For a classical vacuum state corresponding to constant diagonal matrices the bosonic action remains zero under this shift

⁷The Pfaffian is real positive definite for all N in this limit and hence no issue of reweighting occurs

κ	$\langle S_B \rangle$	S_B^{exact}
1.0	4.40(2)	4.5
10.0	4.47(2)	4.5
100.0	4.483(15)	4.5

Table 2: Bosonic action for $SU(2)$ $Q = 4$ model at several couplings

κ	$\langle S_B \rangle$	S_B^{exact}
1.0	11.71(2)	12.0
10.0	11.98(3)	12.0
100.0	11.98(4)	12.0

Table 3: Bosonic action for $SU(3)$ $Q = 4$ model at several couplings

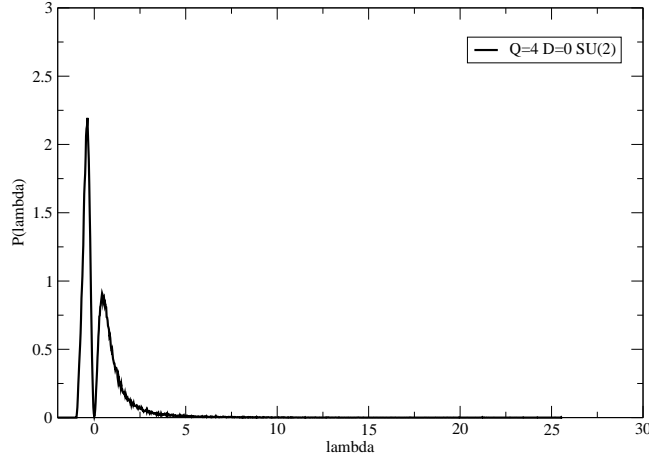


Figure 2: Probability distribution of eigenvalues of $\mathcal{U}_\mu^\dagger \mathcal{U}_\mu - I$ for $Q = 4$ and $SU(2)$ and $D = 0$

Having verified that the breaking of supersymmetry is indeed small after the truncation from $U(N)$ to $SU(N)$ we now turn to the scalar field eigenvalues. Figure 2 shows a plot of the probability distribution of the eigenvalues of $\mathcal{U}_\mu^\dagger \mathcal{U}_\mu - I$ for $SU(2)$ gauge group. Classically the model contains flat directions corresponding to constant diagonal \mathcal{U} matrices. One might worry that the presence of these flat directions might render the path integral defining the quantum theory ill defined but these numerical results, like earlier matrix model studies [33, 34], confirm that the scalar fields are localised around the origin in moduli space and the partition function is finite⁸. The asymmetry in the distribution is a cut-off effect; the

⁸As has been seen before [35] the distribution of scalar field eigenvalues is observed to possess N peaks for

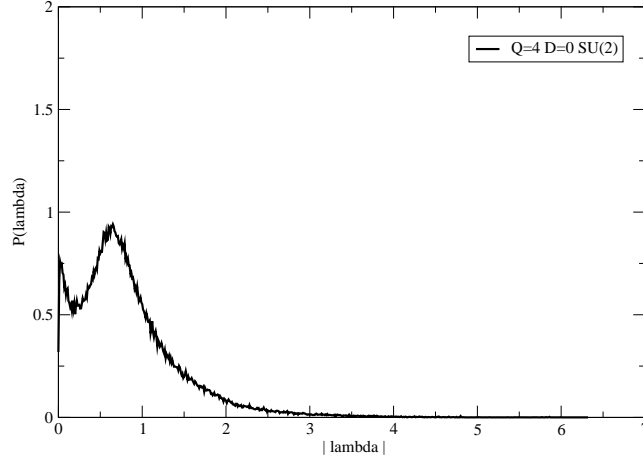


Figure 3: Probability distribution of eigenvalues of the fermion operator for $Q = 4$, $SU(2)$ and $D = 0$

lattice theory, unlike the continuum theory, is not invariant under a change of sign of the scalar fields. Indeed, the entire region $-\infty < B_\mu < 0$ is mapped onto the finite segment $-1 < U_\mu^\dagger U_\mu - 1 < 0$. The asymmetry in the distribution can be hence be used to quantify the magnitude of lattice artifacts. Since the dimensionless eigenvalues contain a factor of the lattice spacing the tails of the distribution contract and are less sensitive to the hard cut-off $\lambda = -1$ as the lattice spacing is reduced.

More importantly, notice that the effect of the classical flat directions is still visible in the presence of a power law tail in the distribution extending out to large (positive) eigenvalue. Both theoretical arguments and numerical results suggest the power $p = 3$ independent of N for this model [34]. Such a power behavior would yield a logarithmically divergent value for $\langle \lambda^2 \rangle$ which means that in any Monte Carlo simulation the scalars spend an appreciable amount of time far the origin in field space. In the background of one of these field configurations the bosonic action will necessarily develop a near zero mode corresponding to translations along this flat direction. By supersymmetry we expect the fermion operator must also then develop a near zero mode. This can be seen explicitly in figure 3 which shows the distribution of the absolute value of the fermion eigenvalue for this system. An enhancement at small eigenvalue is seen consistent with the previous argument.

4.2.2 $Q = 16$ supercharges

In tables 4, 5 we examine the bosonic action for the $Q = 16$ supercharge model in the matrix model limit. Notice that in the case of $Q = 16$ supercharges the fractional deviation of the expectation value from its exact supersymmetric value is smaller for $SU(3)$ than $SU(2)$ at fixed lattice coupling which is consistent with the breaking effect vanishing in the limit $N \rightarrow \infty$. The same effect was not visible however for $Q = 4$ supercharges.

group $SU(N)$

κ	$\langle S_B \rangle$	S_B^{exact}
1.0	13.67(4)	13.5
10.0	13.52	13.5
100.0	13.48(2)	13.5

Table 4: Bosonic action for $SU(2)$ $\mathcal{Q} = 16$, $D = 0$ model at several couplings

κ	$\langle S_B \rangle$	S_B^{exact}	$\langle \cos \alpha \rangle$
1.0	36.01(20)	36.0	0.67(5)
10.0	35.95(18)	36.0	0.47(12)
100.0	35.92(10)	36.0	0.47(13)

Table 5: Bosonic action for $SU(3)$, $\mathcal{Q} = 16$, $D = 0$ model at several couplings

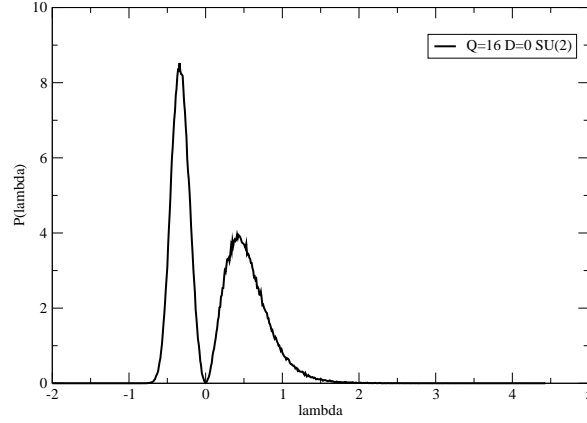


Figure 4: Probability distribution of eigenvalues of $\mathcal{U}_\mu^\dagger \mathcal{U}_\mu - I$ for $\mathcal{Q} = 16$, $SU(2)$ and $D = 0$

Additionally notice that table 5 contains a fourth column corresponding to the average value of the cosine of the Pfaffian phase (the average sine being always consistent with zero). While the Pfaffian for the $\mathcal{Q} = 16$ supercharge model is generically complex it is possible to show that it is real positive definite for the $SU(2)$ model in the matrix model limit. In the case of $SU(3)$ the Pfaffian in zero dimensions is real but not necessarily positive definite and hence the bosonic action is reweighted with the sign of the Pfaffian, as discussed in the previous section.

The probability distribution for $\mathcal{U}_\mu^\dagger \mathcal{U}_\mu - I$ for $\mathcal{Q} = 16$ and $SU(2)$ is shown in figure 4. While it is qualitatively similar to the $\mathcal{Q} = 4$ case it should be clear that the localization around the origin is more dramatic than for $\mathcal{Q} = 4$ – indeed, theory and simulation point to a larger value of the power law exponent $p = 15$ governing the tail of the distribution. Since now the scalar fields do not penetrate far down the classical flat directions one would expect

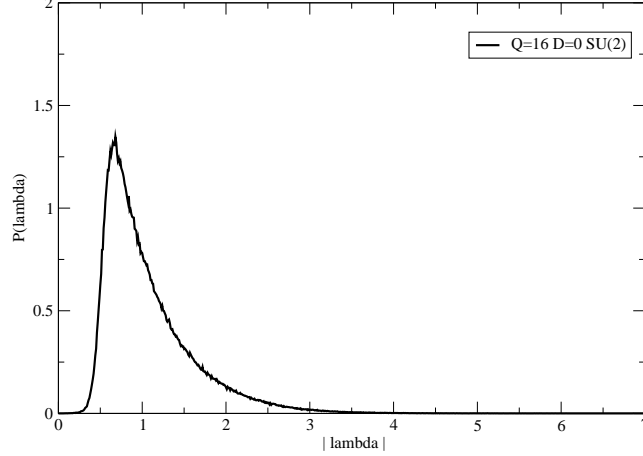


Figure 5: Probability distribution of eigenvalues of the fermion operator for $Q = 16$, $SU(2)$ and $D = 0$

κ	$\langle S_B \rangle^q$	$\langle S_B \rangle$	S_B^{exact}	$\langle \cos \alpha \rangle$
8.0	17.15(1)	17.56(3)	18.0	-0.24(1)
18.0	39.23(2)	41(6)	40.5	-0.06(2)
32.0	70.61(4)	65(5)	72.0	-0.014(6)

Table 6: Observables for $SU(2)$ $Q = 4$ model in $D = 2$

to see correspondingly fewer near zero modes in the fluctuation spectra of either the bosonic or, by supersymmetry, fermionic operator in this case. Figure 5 shows that indeed this is the case – a gap appears to open up in the probability distribution of the magnitude of the fermion eigenvalue close to the origin.

We will see that these qualitative features survive in the two dimensional case to which we now turn.

5. Two dimensions

5.1 $Q = 4$ supercharges

We have also begun an investigation of the $Q = 4$ model with gauge group $SU(2)$ in two dimensions. Again, we have focused on the bosonic action, the distribution of scalar eigenvalues and the spectrum of the fermion operator. In these simulations we hold fixed the continuum dimensionless 't Hooft coupling $\lambda\beta^2 = 0.5$ as defined in eqn. 2.8 which leads to a lattice coupling that grows like the square of the lattice length L . Table 6 shows results from simulations on lattices of size $L = 2, 3, 4$ for gauge group $SU(2)$. The second column corresponds to the phase quenched approximation in which the data is *not* reweighted by the Pfaffian phase. The third column corresponds to the reweighted action. Notice that the phase

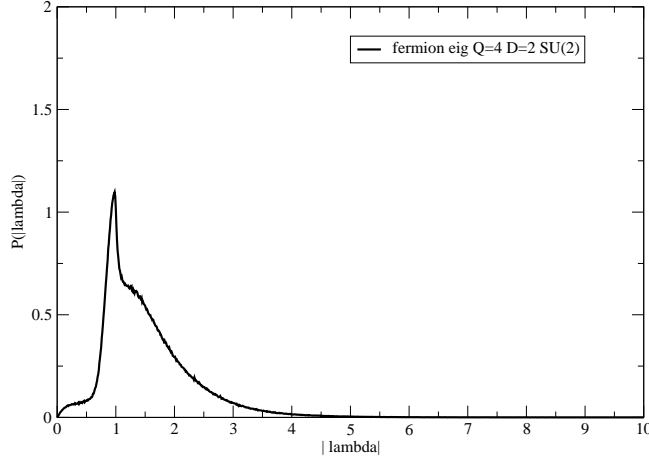


Figure 6: Probability distribution of absolute fermion eigenvalue for $\mathcal{Q} = 4$ $D = 2$ $L = 2$ and $SU(2)$

quenched numbers are relatively close to their exact values. However, while there is a hint from the data that reweighting pushes them closer to those exact values, it should be clear that the statistical errors increase rapidly with lattice size rendering it impossible to make meaningful measurements on the larger lattices. This is confirmed by looking at the average of the cosine of the Pfaffian phase shown in the fifth column. It is statistically consistent with zero on the larger lattices.

Actually, the observation that the phase factor vanishes in this theory is quite interesting; since we employ periodic boundary conditions the measured expectation value yields the Witten index for the theory:

$$W = \langle e^{i\alpha} \rangle_{\text{phase quenched}} \quad (5.1)$$

Thus our data is consistent with a vanishing Witten index – a necessary condition for supersymmetry breaking. Furthermore, as in the matrix model case, we have observed a large number of rather small fermion eigenvalues – figure 6 shows the distribution of the absolute value of the fermion eigenvalue for the $SU(2)$ model on a lattice with $L = 2$. The existence of many small magnitude eigenvalues is highlighted by the scatter plot of figure 7 showing the real and imaginary parts of all fermion eigenvalues with magnitude $|\lambda| < 1.5$ obtained from a sample of 1000 configurations for $L = 2$. Notice that the eigenvalues, while concentrated in a band along the imaginary axis, nevertheless have a non-zero density, by virtue of the Yukawa couplings, over the entire plane including the region around the origin. As we will see later this last feature is *not* seen for $\mathcal{Q} = 16$ supercharges where we observe essentially no eigenvalues in the vicinity of the origin.

For $\mathcal{Q} = 4$ it is possible that these near massless states could play a role as Goldstino modes associated with dynamical supersymmetry breaking which has been conjectured to happen for the $\mathcal{Q} = 4$ theory in low dimensions [36].

The scalar eigenvalue distribution is similar to its cousin in zero dimensions and is shown

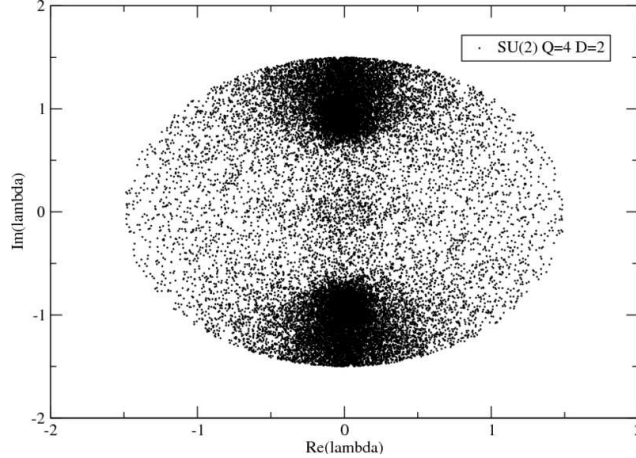


Figure 7: Eigenvalues of the fermion operator for $Q = 4$, $SU(2)$ and $D = 2$

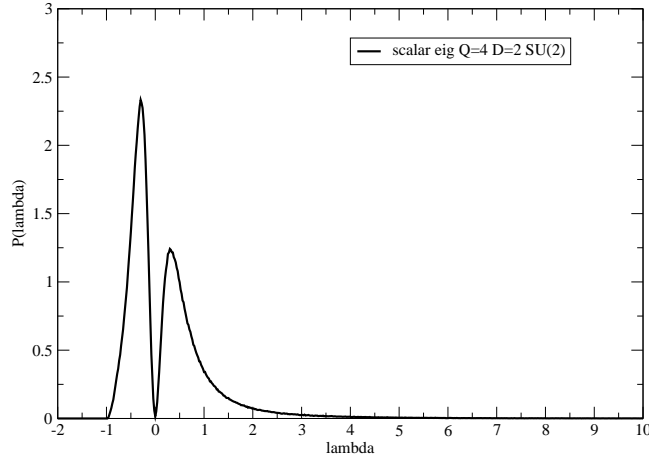


Figure 8: Probability distribution of eigenvalues of $U_\mu^\dagger U_\mu - I$ for $Q = 4$, $SU(2)$ and $D = 2$

in figure 8 and shows again that the scalar distribution possesses a tail to large eigenvalue. Furthermore, we have observed that there is a correlation between the occurrence of small fermion eigenvalues and the presence of large scalar fields associated with a significant component of the scalar fields along the classical flat directions.

Our overall conclusion is that practical simulations of the $Q = 4$ supercharge theory in two dimensions with supersymmetry preserving periodic boundary conditions will be extremely hard due the rapid fluctuations in the phase of Pfaffian - a problem which was first highlighted in [37]. Furthermore, it is possible that these phase variations are associated with a dynamical breaking of supersymmetry as argued for in [36]. Our numerical results support a vanishing Witten index and the presence of massless fermions – necessary conditions for supersymmetry

κ	$\langle S_B \rangle^q$	$\langle S_B \rangle$	S_B^{exact}	$\langle \cos \alpha \rangle$
8.0	53.26(6)	53.26(6)	54.0	0.999997(1)
18.0	120.1(2)	120.1(2)	121.5	0.999995(1)
32.0	214.7(4)	214.6(3)	216.0	0.999994(3)

Table 7: Observables for $SU(2)$ $Q = 16$ model in $D = 2$

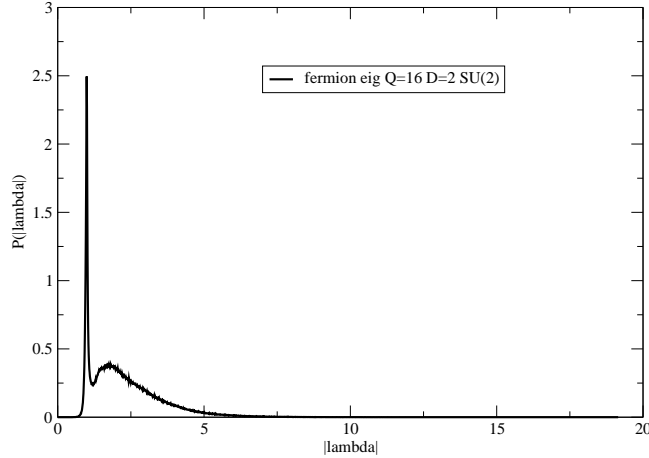


Figure 9: Probability distribution of fermion eigenvalues for $Q = 16$, $D = 2$, $L = 2$ and $SU(2)$

breaking. If so these results contradict the numerical results presented in [31]. A spontaneous breaking of supersymmetry would also invalidate the theoretical arguments given by Matsuura [38] showing that the orbifold theories have zero vacuum energy. However the latter result is derived using semiclassical exactness which is invalid if the Q -supersymmetry is spontaneously broken by non-perturbative effects. Clearly, further work is required to resolve this question unambiguously.

5.2 $Q = 16$ supercharges

The results for the bosonic action for the $Q = 16$ model are shown in table 7 and correspond to simulations on lattices of size $L = 2, 3, 4$ and gauge group $SU(2)$. Perhaps the most striking result is that the Pfaffian phase is very close to zero and remains so as the lattice size is increased and the continuum limit approached. In this case the reweighting procedure is irrelevant as can be seen by comparing the phase quenched and full expectation values shown in table 7.

We believe that this suppression in the Pfaffian phase is related to the phenomena we observed in zero dimensions; the scarcity of small fermion eigenvalues. A plot of the distri-

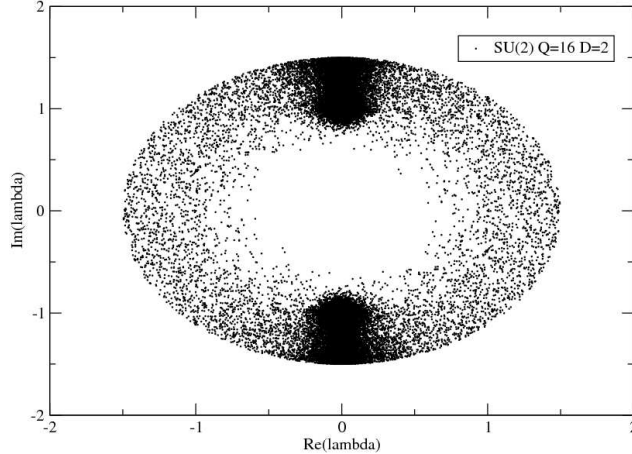


Figure 10: Eigenvalues of the fermion operator for $Q = 16$, $SU(2)$ and $D = 2$

L	$\langle S_B \rangle^q$	$\langle S_B \rangle$	S_B^{exact}	$\langle \cos \alpha \rangle$
2	211.2(2)	211.2(2)	216.0	0.9945(3)
3	1072.8(10)	1075.0(35)	1093.5	0.955(6)

Table 8: Observables for $SU(2)$ $Q = 16$ model in $D = 4$ at $\lambda = 0.5$

bution of the absolute fermion eigenvalue for this theory is shown in figure 9 and indeed a gap appears to open up in the spectrum. This conclusion is reinforced when we examine a scatter plot of the fermion eigenvalues in the complex plane given in figure 10. Remarkably, and in contrast to the $Q = 4$ model, the eigenvalues are excluded from a region around the origin and concentrate along the imaginary axis. It is reasonable to conjecture that it is the small eigenvalues that control the phase; after all in the limit in which the eigenvalues are confined to the imaginary axis an eigenvalue must flow from positive to negative values for a sign change to occur. Such a transition would require a zero eigenvalue to arise which we observe to be highly unlikely. Thus phase fluctuations are suppressed.

Furthermore, the presence of a gap in the fermion spectrum correlates to a rather rapidly damped scalar eigenvalue distribution for the $Q = 16$ model as we saw in zero dimensions. This distribution is shown in figure 11.

6. Four dimensions

Finally we report on our preliminary simulations of the $Q = 16$ supercharge theory in four dimensions. Table 8 shows the bosonic action and cosine of the Pfaffian phase for lattices with size $L = 2, 3$ at fixed 't Hooft coupling $\lambda = 0.5$ (the data corresponds to 6000 and 1000 configurations for $L = 2$ and $L = 3$ respectively)

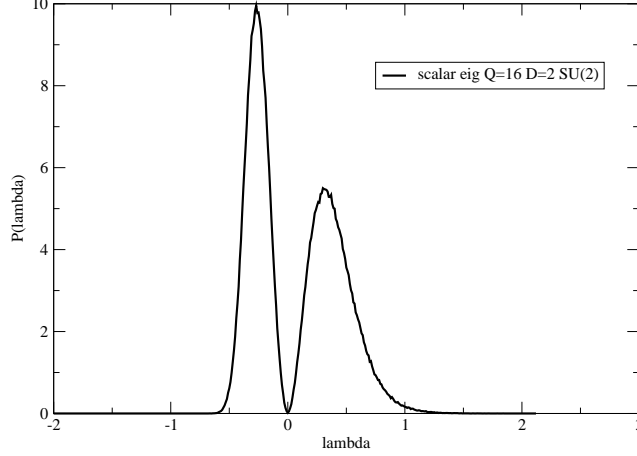


Figure 11: Probability distribution of $\mathcal{U}_\mu^\dagger \mathcal{U}_\mu - I$ eigenvalues for $\mathcal{Q} = 16$, $D = 2$, $L = 2$ and $SU(2)$

L	$\langle S_B \rangle^q$	$\langle S_B \rangle$	S_B^{exact}	$\langle \cos \alpha \rangle$
2	212.5(3)	211.5(5)	216.0	0.9964(2)
3	1081.9(17)	1080.5(45)	1093.5	0.983(2)

Table 9: Observables for $SU(2)$ $\mathcal{Q} = 16$ model in $D = 4$ at $\lambda = 0.25$

While supersymmetry breaking effects are visible at $O(1)\%$ the Pfaffian phase is small and reweighting offers a reliable way to deal with the fluctuations at least for small lattices⁹. It remains to be seen whether this is true as the lattice size is increased but we see these results as encouraging (after all we have lost control of the phase already with small lattices for the $\mathcal{Q} = 4$ model).

For comparison, table 9 shows the same quantities for 't Hooft coupling $\lambda = 0.25$. Notice that as we approach weak coupling and smaller lattice spacings the bosonic action moves towards its exact supersymmetric value as expected. Furthermore, the phase fluctuations also decrease. The question of whether reweighting will be practical on larger lattices will hence depend on which trajectory $\lambda_{\text{bare}}(L)$ we must follow in the (λ, L) plane to approach the continuum limit.

The scalar and fermion eigenvalue distributions are shown in figures 12 and 13 and look qualitatively similar to their two dimensional cousins.

Again, the scatter plot of fermion eigenvalues also indicates that eigenvalues are repelled from a region around the origin as for the $\mathcal{Q} = 16$ theory in two dimensions.

⁹The larger error in the reweighted numbers for $L = 3$ reflects only the decreased statistics available in that case where the Pfaffian is only computed every tenth measurement

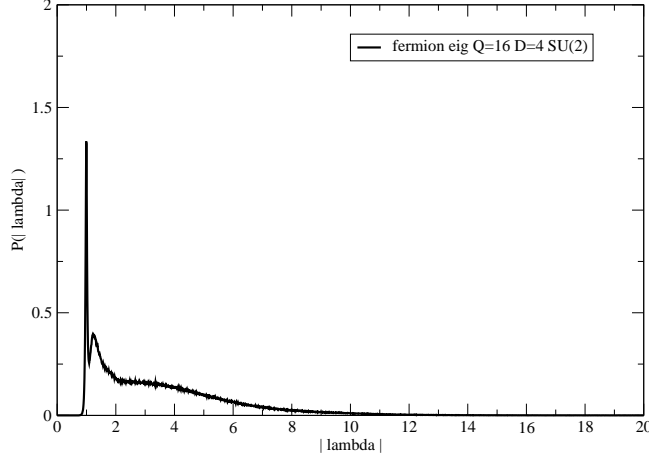


Figure 12: Probability distribution of fermion eigenvalues for $Q = 16$, $D = 4$, $L = 2$ and $SU(2)$

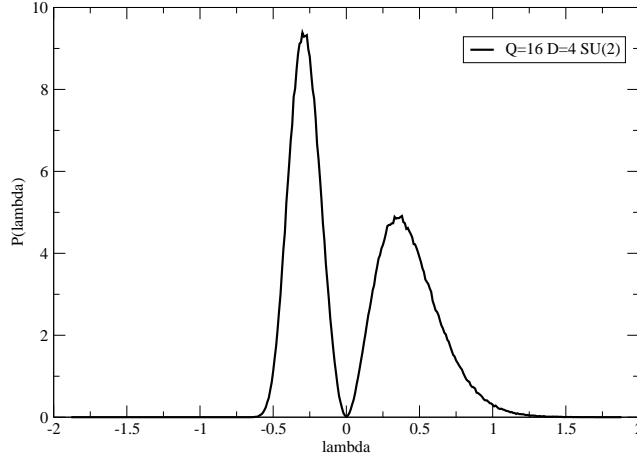


Figure 13: Probability distribution of scalar eigenvalues for $Q = 16$, $D = 4$, $L = 2$ and $SU(2)$

7. Discussion

In this paper we have presented numerical results from simulations of a variety of Yang-Mills theories with extended supersymmetry. The lattice actions that are employed can be derived either with orbifold methods [1, 2, 3] or via geometrical discretization of a twisted version of the target theory [8]. Remarkably, they possess both gauge invariance and one more exact supersymmetries at non-zero lattice spacing. These lattice theories are important both at a conceptual and practical level; they offer the possibility of a rigorous definition of the continuum gauge theory, and through numerical simulation may offer up new ways to extract non-perturbative information on that gauge theory. In this paper we have focused on

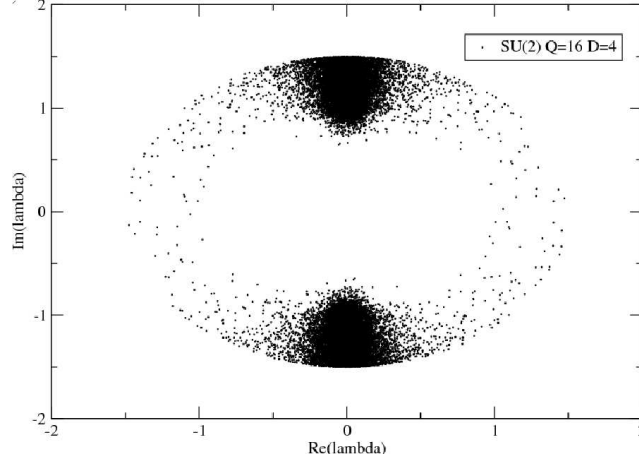


Figure 14: Eigenvalues of the fermion operator for $Q = 16$, $D = 4$ and $SU(2)$

the latter question; specifically, are these lattice theories amenable to Monte Carlo simulation using the tools and techniques of lattice gauge theory ?

We have shown that the $U(N)$ theories generically suffer from a vacuum stability problem – the theory is defined in terms of complexified Wilson gauge links and these develop one or more zero eigenvalues under quantum corrections. This instability is associated with the trace mode of the would be scalar fields in the lattice theory. As a result the link fields are driven a long way from the identity for any value of the lattice spacing rendering invalid the correspondence between the lattice model and the target continuum theory.

However we have shown that this instability of the lattice action is avoided if the theory is truncated to $SU(N)$ ¹⁰. Furthermore, by measurement of a simple supersymmetric Ward identity we have presented evidence that the associated breaking of supersymmetry is small and decreases as the continuum limit is taken.

A second potential problem arises however; integration over the fermions in these models generically leads to a *complex* Pfaffian (which can be reduced to a determinant for $Q = 4$ models). Our simulations are necessarily performed in the phase quenched ensemble where this phase is ignored. In principle, expectation values can be computed in the full ensemble by a reweighting procedure. However, this is only practical if the phase fluctuations are small. In the case of the $Q = 4$ theory we find the average phase factor approaches zero rapidly as the continuum limit is taken and the statistical errors in the reweighted observables grow uncontrollably. This seems to rule out the possibility of using these actions to study the $Q = 4$ theory at least for zero temperature (periodic temporal boundary conditions for all fields).

Intriguingly, the vanishing expectation value of the phase factor $\langle e^{i\alpha} \rangle_{\text{phase quenched}} = 0$ may be given a physical interpretation - it corresponds naively to a vanishing Witten index

¹⁰this truncation also has the merit of removing an exact zero mode of the fermion operator

for the model. Of course the notion of a Witten index as a signed sum over classical vacua is somewhat delicate in theories with extended supersymmetry which possess a continuum of such vacua. Nevertheless, a vanishing Witten index is a necessary condition for supersymmetry breaking and our analysis of the fermion spectrum shows that indeed this theory contains one or more near zero modes which could play the role of a Goldstino associated with dynamical breaking of supersymmetry. An argument for such a breaking has been independently by Hori et al [36] for the $\mathcal{Q} = 4$ theory in two dimensions.

Furthermore, we have observed that the appearance of small fermion eigenvalues is correlated with large excursions of the scalar fields along the classical directions. In the background of such a vacuum configuration the operator describing small fluctuations of the scalars develops a small eigenvalue corresponding to motion along the flat direction. This eigenvalue is mirrored in the fermionic sector because of supersymmetry. The question of whether this effect can lead to supersymmetry breaking is then tied to the question of how frequently, in the context of a Monte Carlo simulation, the scalars probe these flat directions which, in turn, is measured by the probability distribution of scalar field eigenvalues. For $\mathcal{Q} = 4$ we observe that this distribution $P(\lambda)$ shows a slow power law decay which has been estimated to vary as λ^{-3} for large λ [33, 34]. This could supply the effect we have argued for; the variance of λ then diverges logarithmically and the typical scalar field configuration wanders appreciably away from the origin and leads to a non-zero density of small fermion eigenvalues.

At first sight the strong phase fluctuations visible in the lattice $\mathcal{Q} = 4$ theory are surprising since a proof exists that the continuum Pfaffian is real positive semi-definite and one might have therefore expected the phase fluctuations to be suppressed for small enough lattice spacing. This does not appear to be the case. However, we believe there is no inconsistency; if supersymmetry breaks spontaneously as we claim, the Pfaffian will be *zero* on the important continuum field configurations dominating the path integral, which is then consistent with the vanishing of the expectation value of the phase factor seen in the lattice simulations.

In the case of the $\mathcal{Q} = 16$ supercharge model the corresponding scalar distribution falls much more rapidly with eigenvalue. Correspondingly, we observe that the fermion spectrum shows a complete absence of near zero modes and the Pfaffian phase is typically small and can be handled, at least for these small lattices and for $SU(2)$, by reweighting. This is naively rather unexpected – afterall the Pfaffian is generically complex for the $\mathcal{Q} = 16$ model. However, as we have argued previously, the average phase factor in the phase quenched ensemble is nothing more than the Witten index of the model which is a topological invariant. It can hence be evaluated *exactly* in the semi-classical limit along the lines described in [38] provided supersymmetry remains unbroken which is thought to be the case for the $\mathcal{Q} = 16$ model. Such a calculation shows that $W = 1$. Thus topological arguments would suggest that it should be possible to handle the Pfaffian phase for the $\mathcal{Q} = 16$ theory using reweighting techniques even though the Pfaffian of the theory is generically complex. This is a remarkable result but quite consistent with the results of our simulations.

The absence of small fermion eigenvalues together with the seeming relative unimportance of Pfaffian phase fluctuations lead us to feel cautiously optimistic about the feasibility of

conducting large scale simulations of the $\mathcal{Q} = 16$ supercharge theory in four dimensions in the near future. Such simulations are of course very interesting from the point of view of the AdSCFT correspondance and we hope to report on results from such simulations in the near future [39].

Acknowledgments

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